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Search for minimal quantum group $SU_q(2)$ gauge field theory

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Different possibilities for the introduction of quantum group gauge fields are discussed. The case of the quantum group $SU_q(2)$ is considered in more detail. We seek for a construction of the quantum group gauge fields which possesses a minimal set of usual c-number fields. It turns out that in this construction the components of the quantum group gauge field take values in the quantum Euclidean space.

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1. Introduction

Quantum groups have already manifested themselves as one of the branches of mathematical physics and they give a powerful tool in investigations of integrable systems, conformal field theory and the theory of braids [1–5].

One can also use quantum groups to construct a new class of models with fields taking values in quantum spaces, quantum groups or quantum algebras [6,7]. This idea is related to the treatment of quantum groups as q-deformations of usual groups. It is tempting to speculate that q-deformations of usual groups could be applied in the building of realistic models. Many geometrical and algebraic notions of classical groups admit q-deformations. In particular, quantum groups can be viewed as symmetries of "q-deformed" vector spaces [1,5] and there exists an analog between differential calculus on quantum spaces and on quantum groups [4,8].

There are two kinds of quantum group field model. One deals with the usual space–time variables (or space–time lattice), the other is based on noncommutative space–time. A simple example of a quantum group dynamic system is classical mechanics on a quantum line [9]. The corresponding quantum theory is q-deformed quantum mechanics [9,10]. The consideration of noncommutative space–time is related to the q-deformed Lorentz [11–13] and Poincaré [14] groups.

In this paper we would like to discuss the idea of constructing quantum group gauge field theories on classical space–time [6]. This is a rather promising proposal and apparently there are several possibilities to realize it. The general point consists in making gauge transformations belonging to some quantum group. Along this line one may think of *global* as well as *local* transformations. We will consider some general aspects of global transformations.

The major question arising in treating a quantum group as a symmetry group is to understand a sequence of gauge transformations. The usual product of two arbitrary transformations is not an element of the quantum group or, in other words, the quantum group is not a group in the usual sense. However, due to the existence of the Hopf algebra structure on quantum groups a natural answer consists in considering a special class of transformations. Each of these transformations can be seen as a matrix with noncommuting entries belonging to some algebra $\mathcal{A}$. Then the product of two transformations defines an element of the quantum group if all entries of one commute with all entries of the other.

For example, an element of the quantum group $SU_q(2)$ is a 2×2 matrix of the canonical form:

$$g = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \quad a, c \in \mathcal{A}, \tag{1}$$

with the unitarity condition

$$gg^* = g^*g = I. \tag{2}$$

From the unitarity condition one gets the commutation relations between the elements a and c (see appendix A). Then two $SU_q(2)$-transformations

$$g = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} a' & -qc'^* \\ c' & a'^* \end{pmatrix}$$

with the commutation relations (A.1)–(A.5) give the $SU_q(2)$ transformation $g'' = gg'$ if

$$[g_{ij}, g'_{kl}] = 0 \ . \tag{3}$$

The simplest way of realizing requirements (3) is to consider the tensor product of g and g'.

Another question one has to answer is the choice of gauge fields. This is a nontrivial question since the relation between Lie algebras and Lie groups becomes more involved for quantum groups. As was mentioned in ref. [6] there are at least two different possibilities to introduce the quantum group gauge field. Roughly speaking, the first one is closely related with the q-deformed universal enveloping algebra, the second is based on differential calculus on quantum groups. Note that only the gauge field in a certain gauge is meaningful. In the first case the field A_μ must have nontrivial commutation relations with group elements and in the second one we should assume that $[g_{ij}, A_{kl}] = 0$. But in both cases one cannot think of gauge fields as independent objects, since they have to be attached to some gauge. Let us view these two possibilities in more detail.

1. *The gauge fields take values in an analog of the Lie algebra for quantum groups.* An appropriate definition based on the exponential map was considered in ref. [6] for the quantum group $SU_q(2)$. In this construction the gauge potential has the form

$$A_\mu(x) = l^i_\mu(x) \otimes E_i,$$

where $\{E_i\}$ is the basis of the q-deformed universal enveloping algebra, l^i_μ are the subject of the quantum superplane and the commutation relations between the elements of the quantum group and l^i_μ are nontrivial [7]. To generalize this construction to other groups one has to build exponential maps on them. In section 4 we present the exponential map for the quantum group $SO_q(3, \mathbb{C})$.

2. *The gauge field takes values in the q-deformed universal enveloping algebra* $U_q(g)$ *of a Lie algebra g*:

$$A_\mu(x) = \sum_{i_1, i_2, \ldots, i_n, \ldots} A_\mu(x)^{i_1 i_2 \cdots i_n} t_{i_1} \cdots t_{i_n} , \tag{4}$$

where t_i belong to the q-deformed universal enveloping algebra. In comparison with the usual case, when A_μ is an element of the Lie algebra, we have an infinite number of component fields $A_\mu(x)^{i_1 i_2 \cdots i_n}$. However, in a given representation of the q-deformed universal enveloping algebra one need not keep the whole tower in formula (4). For example, in the case of the fundamental representation of $U_q(\mathrm{su}(2))$ it is enough to consider four-component fields:

$$A_\mu(x) = A^i_\mu(x) \otimes \sigma_i + A^0_\mu(x) \otimes I. \tag{5}$$

In (5) the fact is used that the fundamental representation of the q-deformed universal enveloping algebra of su(2) can be written in terms of the σ-matrices. It is evident that the usual Lagrangian $\mathcal{L} = \mathrm{tr}\, F^2_{\mu\nu}$ is invariant under infinitesimal transformations with gauge parameter decomposed as in (5). Thus one gets the usual $SU(2) \otimes U(1)$ gauge theory with one coupling constant.

One can go further and consider global finite transformations belonging to the quantum group. Performing a global $SU_q(2)$ transformation we see that one cannot let A^i_μ be c-numbers and one has to assume that the components A^i_μ belong to some algebra $\mathcal{B}$. For example, it is possible to consider a free algebra generated by the elements a, a^*, c and c^* modulo relations (A.1)–(A.5) (see appendix A) as $\mathcal{B}$. However, such a construction has an infinite number of usual c-component fields. So it is natural to try to find some minimal set $\mathcal{B}_1$, which is invariant under gauge transformations generated by the elements of the quantum group. One such construction has been proposed by Isaev and

Popovicz [15], who considered the gauge field A_μ of the form (5) with gauge potentials A^i_μ obeying the Witten-type quantum algebra $U_q(su(2))$ [16]. To write down a Lagrangian which is invariant under $SU_q(2)$ transformations they introduced the notion of the quantum trace. Generally speaking the Lagrangian is a combination of two terms,

$$c_1 \operatorname{tr}_q F^2_{\mu\nu} + c_2 (\operatorname{tr}_q F_{\mu\nu})^2 \ . \tag{6}$$

Since $\operatorname{tr}_q F_{\mu\nu} \neq 0$ one gets a theory with two coupling constants. The presence of the second term may be connected with the Weinberg angle θ and $\theta = (q^{-1} - q)/(q^{-1} + q)$ [15].

Unfortunately, the model (6) does not provide the usual Yang–Mills theory in the classical limit $q \to 1$. To guarantee the right classical limit one can slightly modify the representation (5) and consider the case when instead of σ-Pauli matrices the generators E_i, $i = 1, 2, 3$, of the q-deformed universal enveloping algebra of su(2) in the Witten form [16] (see (8)) are used.

In this paper we will show that in some sense a minimal construction consists considering the gauge potential

$$A_\mu(x) = A^i_\mu(x) \otimes E_i, \tag{7}$$

where A^i_μ are elements of the quantum Euclidean space.

This paper is organized as follows. We start with a discussion of algebras of gauge potentials that are invariant under global $SU_q(2)$ transformations. In section 3 we find a connection between $SL_q(2)$ and $SO_{q^2}(3)$. The exponential map for $SO_{q^2}(3, \mathbb{C})$ acting on the quantum Euclidean space is presented in section 4. The commutation relations for the quantum groups $SU_q(2)$, $SL_q(2)$ and $SO_q(3, \mathbb{C})$ are collected in the appendices.

2. The algebra invariant under global $SU_q(2)$ transformations

We are going to consider in this section the quantum group $SU_q(2)$. However, for the sake of generality, we deal with $SL_q(2)$, having in mind that the reduction to the case of $SU_q(2)$ is allowed at any moment since $SU_q(2)$ is the real form of $SL_q(2)$. We start with the noncommutative algebra generated by 1 and three symbols E_0, E_+, E_- modulo the relations

$$\begin{aligned} qE_0E_+ - \frac{1}{q}E_+E_0 &= \mu^{+2}_{1/2}E_+ \, , \\ \frac{1}{q}E_0E_- - qE_-E_0 &= -\mu^{+2}_{1/2}E_- \, , \\ [E_+, E_-] &= (\mu^+_{1/2}/\mu)(\mu^+_{1/2}E_0 - \mu^-_{1/2}E_0^2) \, , \end{aligned} \tag{8}$$

where q is a complex parameter, $\mu^\pm_{1/2} = q^{1/2} \pm q^{-1/2}$, $\mu = q + q^{-1}$. After the appropriate rescaling of the generators E_0, E_+, E_- these relations reproduce the Witten-type algebra $U_q(\mathrm{sl}(2))$ [16]. So in the following we shall refer to the relations (8) as the quantum algebra $U_q(\mathrm{sl}(2))$. (Note that relations (8) are not isomorphic to those of the standard q-deformed universal enveloping algebra [16].)

As was shown by Isaev and Popovicz [15], the quantum algebra (8) is invariant under the transformation $E_i \to E_i'$ generated by the formula

$$\mathbf{E} \longrightarrow g\mathbf{E}g^{-1}, \tag{9}$$

where

$$\mathbf{E} = \begin{pmatrix} (q/\mu)E_0 & E_+ \\ E_- & -(1/q\mu)E_0 \end{pmatrix}, \tag{10}$$

or

$$\begin{pmatrix} E_0' \\ E_+' \\ E_-' \end{pmatrix} = \frac{1}{\det_q g}\begin{pmatrix} \det_q g + \mu bc & -\mu ac & \mu db \\ -ba & a^2 & -(1/q)b^2 \\ cd & -qc^2 & d^2 \end{pmatrix}\begin{pmatrix} E_0 \\ E_+ \\ E_- \end{pmatrix}. \tag{11}$$

Here $\det_q g = 1$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_q(2)$, and all E_i commute with the entries of g. Now we are going to consider the transformation (9) at the level of the peculiar matrix representation of $U_q(\mathrm{sl}(2))$. The quantum algebra $U_q(\mathrm{sl}(2))$ has the representation by 2×2-matrices:

$$E_0 = k\begin{pmatrix} 1 & 0 \\ 0 & -1/q^2 \end{pmatrix}, \quad E_+ = \frac{k}{q}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \frac{k}{q}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

where $k = (1 + 1/q^4)^{-1}(1 + 1/q)^2$. E_0, E_+, E_- transform under (11) as

$$E_0' = \frac{k}{\det_q g}\begin{pmatrix} \det_q g + \mu bc & -(1/q)\mu ac \\ -(1/q)\mu db & -(1/q^2)(\det_q g + \mu bc) \end{pmatrix},$$

$$E_+' = \frac{k}{\det_q g}\begin{pmatrix} -ba & (1/q)a^2 \\ -(1/q^2)b^2 & (1/q^2)a^2 \end{pmatrix},$$

$$E_0' = \frac{k}{\det_q g}\begin{pmatrix} cd & -c^2 \\ (1/q)d^2 & -(1/q^2)cd \end{pmatrix}. \tag{13}$$

The matrices E_i' (if $\det_q g = 1$) also give the representation of $U_q(\mathrm{sl}(2))$ but their entries are no longer numbers. One can find the following

Proposition 1. *There exists an element $O \in \mathrm{GL}_q(2)$ such that the transitions (13) can be represented in the form*

$$OE_iO^{-1} = \kappa_i E_i'.$$

There is no summation in this formula and $\kappa_0 = 1$, $\kappa_+ = q$, $\kappa_- = 1/q$.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_q(2)$. Introduce a matrix O

$$O = \|O_{ij}\| = \begin{pmatrix} \sqrt{q}a & \sqrt{q}c \\ (1/\sqrt{q})b & (1/\sqrt{q})d \end{pmatrix}.$$

If a, b, c, d obey the commutation relations defining $\mathrm{GL}_q(2)$ then the entries O_{ij} also obey them. In addition we have

$$\det_q O = O_{11}O_{22} - qO_{12}O_{21} = ad - qbc = \det_q g\,.$$

This means that $O \in \mathrm{GL}_q(2)$ and the transition from g to O is an automorphism of $\mathrm{GL}_q(2)$. The inverse of O has the form

$$O^{-1} = \frac{1}{\det_q O}\begin{pmatrix} (1/\sqrt{q})d & -(1/\sqrt{q})c \\ -\sqrt{q}b & \sqrt{q}a \end{pmatrix}.$$

Taking into account that $\det_q O$ is central and using the commutation relations for $\mathrm{GL}_q(2)$ (see appendix A), one can check that

$$\begin{aligned} OE_0O^{-1} &= \frac{1}{\det_q O} \\ &\times \begin{pmatrix} \sqrt{q}a & \sqrt{q}c \\ (1/\sqrt{q})b & (1/\sqrt{q})d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1/q^2 \end{pmatrix}\begin{pmatrix} (1/\sqrt{q})d & -(1/\sqrt{q})c \\ -\sqrt{q}b & \sqrt{q}a \end{pmatrix} \\ &= E'_0\,. \end{aligned}$$

In the same manner we establish

$$OE_+O^{-1} = qE'_+\,, \qquad OE_0O^{-1} = (1/q)E'_-\,.$$

The proof is complete. □

Let A be a field taking values in the 2×2-matrix representation of $U_q(\mathrm{sl}(2))$ in the form (12):

$$A(x) = A^0(x)\otimes E_0 + A^+(x)\otimes E_+ + A^-(x)\otimes E_-\,.$$

Here the tensor product means that $[A^i, O_{kl}] = 0$. Then the existence of the matrix O enables us to consider global transformations generated by the elements of $\mathrm{SL}_q(2)$:

$$\begin{aligned} A(x) \longrightarrow OA(x)O^{-1} &= A^i(x)\otimes OE_iO^{-1} \\ &= A^0(x)\otimes E'_0 + qA^+(x)\otimes E'_+ + (1/q)A^-(x)\otimes E'_-\,. \end{aligned} \tag{14}$$

Using the expansion of the basis E'_i over E_i:

$$E'_i = c_{ij}(a,b,c,d)E_j\,,$$

we get the following transformation of the potentials A^i under (14):

$$\begin{pmatrix} A^{0'} \\ A^{+'} \\ A^{-'} \end{pmatrix} = \frac{1}{\det_q g}\begin{pmatrix} \det_q g + \mu bc & -qba & (1/q)cd \\ -\mu ac & qa^2 & -c^2 \\ \mu db & -b^2 & (1/q)d^2 \end{pmatrix}\begin{pmatrix} A^0 \\ A^+ \\ A^- \end{pmatrix}. \tag{15}$$

To give a meaning to the transformation (15) we have to specify a set of fields A^i that would be invariant under the transformation (15). The simplest answer is given by

Proposition 2. *The relations*

$$\mu(qA^0A^+ - (1/q)A^+A^0) = \nu\mu^{+2}_{1/2}A^+ ,$$
$$\mu((1/q)A^0A^- - qA^-A^0) = -\nu\mu^{+2}_{1/2}A^- ,$$
$$A^+A^- - A^-A^+ = \nu\mu^{+2}_{1/2}A^0 - \mu^-\mu A^{0\,2} , \qquad (16)$$

where ν is a complex parameter, $\mu^- = q - 1/q$, are invariant under the transformation (15) with $\det_q g = 1$.

The proof of proposition 2 can be obtained by straightforward calculations. Recall that all A^i commute with the entries of $g \in SL_q(2)$.

Of course, for any nonzero ν the relations (15) are isomorphic to relations (7) of the quantum algebra $U_q(sl(2))$. However, the relations (16) in contrast to (5) allow one to describe a "smooth" limit $\nu \to 0$, which produces the relations

$$A^+A^0 = q^2A^0A^+ , \qquad A^0A^- = q^2A^-A^0 ,$$
$$[A^+, A^-] = (1/q^2 - q^2)(A^0)^2 . \qquad (17)$$

The invariance of (17) with respect to (15) holds and we can treat the potentials A^i as complex numbers in the "classical" limit, $q \to 1$. So in the case $q = 1$ we have ordinary global transformations in the Yang–Mills theory.

We would like to stress that the use of 2 × 2-matrices E_i instead of the Pauli matrices ($a = 1, 2, 3$) allows one to consider the global transformations without involving the field $A^0_\mu \otimes 1$.

To construct an object which would be invariant under global $SU_q(2)$ transformation we need the notion of q-trace [1,15]. If I_q is the matrix $I_q = \begin{pmatrix} 1/q & 0 \\ 0 & q \end{pmatrix}$ then for any matrix **A** the q-trace, Tr_q, is defined by the formula

$$\mathrm{Tr}_q \mathbf{A} = \mathrm{Tr}(I_q\mathbf{A}) = (1/q)A_{11} + qA_{22} .$$

If $[g_{ij}, A_{kl}] = 0$ for any i, j, k, l, then the main property of the q-trace is the following:

$$\mathrm{Tr}_q \mathbf{A} = \mathrm{Tr}_q\, g\mathbf{A}g^{-1},$$

or, in other words, the q-trace is invariant under local transformations generated by the elements $g \in SL_q(2)$. One can see that under a local transformation the tensor $F_{\mu\nu}$ transforms as

$$F_{\mu\nu} \to gF_{\mu\nu}g^{-1}. \qquad (18)$$

Using the property of the q-trace we find two invariants under the transformation (18), $\mathrm{Tr}_q F_{\mu\nu}$ and $\mathrm{Tr}_q F^2_{\mu\nu}$. When $q \neq 1$ and the potentials A^i_μ are non-commuting, the former term,

$$\mathrm{Tr}_q F_{\mu\nu} = \frac{\mu}{q^2}[A^0_\mu, A^0_\nu] + \frac{1}{q^3}(A^+_{[\mu}A^-_{\nu]} - q^2A^-_{[\nu}A^+_{\mu]}),$$

is non-zero and we have a theory with two coupling constants.

3. The quantum Euclidean space

Now we are going to connect the relations (17) with the quantum Euclidean space $O_q^3(\mathbb{C})$, which is invariant with respect to the action of the quantum group $SO_q(3,\mathbb{C})$.

Let us briefly recall the main definitions related with $SO_q(3,\mathbb{C})$ [1]. The R_q-matrix associated with the quantum group $B_1 = SO_q(3)$ has the form

$$R_q^{B_1} = q\sum_{i\neq i'} e_{ii}\otimes e_{ii} + e_{22}\otimes e_{22} + \sum_{i\neq j,j'} e_{ii}\otimes e_{jj} + \frac{1}{q}\sum_{i\neq i'} e_{i'i'}\otimes e_{ii}$$
$$+ (q-1/q)\sum_{i>j} e_{ij}\otimes e_{ji} - (q-1/q)\sum_{i>j} q^{\rho_i-\rho_j} e_{ij}\otimes e_{i'j'},$$

where $i = 1,2,3$, $e_{ij} = \|e_{ij}\|_{kl} = \delta_i^k\delta_j^l$, $i' = 4-i$, $j' = 4-j$, $\rho_1 = \frac{1}{2}$, $\rho_2 = 0$, $\rho_3 = -\frac{1}{2}$. An explicit form of $R_q^{B_1}$ as an element of $M_{3\times 3}(\mathbb{C})$ is

$$R_q^{B_1} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu/\sqrt{q} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mu\mu_{1/2}^{-}/\sqrt{q} & 0 & -\mu/\sqrt{q} & 0 & 1/q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

$R_q^{B_1}$-matrices obey the relations:

$$R_q = C_1(R_q^{T_1})^{-1}C_1^{-1} = C_2(R_q^{-1})^{T_2}C_2^{-1}. \tag{19}$$

Here for a matrix acting on $\mathbb{C}^3\otimes\mathbb{C}^3$ the symbols T_1 and T_2 mean transposition in the first and in the second term, respectively, $C_1 = C\otimes I$, $C_2 = I\otimes C$, and C is a matrix of the form

$$\begin{pmatrix} 0 & 0 & 1/\sqrt{q} \\ 0 & 1 & 0 \\ \sqrt{q} & 0 & 0 \end{pmatrix}.$$

The quantum group commutation relations are given according to the general construction [1]:

$$R(g\otimes 1)(1\otimes g) = (1\otimes g)(g\otimes 1)R, \tag{20}$$

and their explicit forms are collected in appendix B. Equations (19) allow one to impose the additional relations on the entries of the matrix g:

$$gCg^{T}C^{-1} = Cg^{T}C^{-1}g = 1. \tag{21}$$

It is suitable for field theory applications [6] to assign an element of $SO_q(3)$ as a matrix g with entries g_{ij} belonging to some non-commutative algebra $\mathcal{A}$ with quadratic relations (20), (21).

The quantum Euclidean space $O_q^3(\mathbb{C})$ [1] can be treated as a set of three operators

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with the commutation relations:

$$\begin{aligned} x_1 x_2 &= q x_2 x_1 , \qquad x_2 x_3 = q x_3 x_2 , \\ x_1 x_3 - x_3 x_1 &= (1/\sqrt{q} - \sqrt{q})\, x_2^2 . \end{aligned} \tag{22}$$

The natural matrix action of $SO_q(3,\mathbb{C})$ on the quantum Euclidean space $O_q^3(\mathbb{C})$ preserves the relations (22) and, moreover, the requirement of preserving the relations (22) together with the corresponding relations for the differential forms can be treated as a determining property of $SO_{q^2}(3,\mathbb{C})$ [1,5]. Turning back to (15) after the identification

$$x_1 = (1/\sqrt{\mu})A^+ , \qquad x_2 = A^0 , \qquad x_3 = (1/\sqrt{\mu})A^- , \tag{23}$$

and changing $q \to q^2$ in (22), we see that the relations (17) coincide with (22). In terms of x_i, the transformation (15) ($\det_q g = 1$) becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} qa^2 & -\sqrt{\mu}ac & -c^2 \\ -q\sqrt{\mu}ba & 1+\mu bc & (1/q)\sqrt{\mu}cd \\ -b^2 & \sqrt{\mu}db & (1/q)d^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , \tag{24}$$

and we may assume that the matrix

$$P = \begin{pmatrix} qa^2 & -\sqrt{\mu}ac & -c^2 \\ -q\sqrt{\mu}ba & 1+\mu bc & (1/q)\sqrt{\mu}cd \\ -b^2 & \sqrt{\mu}db & (1/q)d^2 \end{pmatrix} \tag{25}$$

belongs to $SO_{q^2}(3,\mathbb{C})$. This ensures the validity of

Proposition 3. *The matrix P of the form (25) with a, b, c, d being as described in appendix B belongs to* $SO_{q^2}(3,\mathbb{C})$.

Proof. Using the commutation relations from appendix A one can easily check the fulfillment of the commutation relations for $SO_{q^2}(3,\mathbb{C})$ (see appendix B) and the additional relations (21). □

Thus on the one hand (24) is the action of $SO_{q^2}(3,\mathbb{C})$ on the quantum Euclidean space and on the other it can be considered as the adjoint representation of $SL_q(2,\mathbb{C})$. Therefore we have in some sense the q-analog of the classical isomorphism:

$$SL(2,\mathbb{C}) \approx SO(3,\mathbb{C})/\mathbb{Z}_2 . \tag{26}$$

If q is real then for $SL_q(2,\mathbb{C})$ and $SO_{q^2}(3,\mathbb{C})$ there exist $*$-involutions defining the real forms, $SU_q(2)$ and $SO_{q^2}(3,\mathbb{R})$, accordingly. Note that the found embedding $SL_q(2,\mathbb{C}) \subset SO_{q^2}(3,\mathbb{C})$ is not compatible with the $*$-involutions and therefore breaks down for the real forms.

Concluding this section we would like to point out a connection of Tr_q and a metric on the quantum Euclidean space [1] of gauge potentials.

The explicit form of the gauge field A_μ is

$$\mathbf{A}_\mu = A^i_\mu \otimes E_i = \begin{pmatrix} A^0_\mu & (1/q)A^+_\mu \\ (1/q)A^-_\mu & -(1/q^2)A^0_\mu \end{pmatrix} .$$

Then we have

$$\mathrm{Tr}_q\, \mathbf{A}_\mu = 0 ,$$
$$\mathrm{Tr}_q\, \mathbf{A}^2_\mu = (\mu/q^2) L_\mu ,$$

where $L_\mu = A^-_\mu A^+_\mu + (1/q^2)(A^0_\mu)^2$ and the relations (19) were used. On the other hand, introducing a vector

$$Y_\mu = \begin{pmatrix} A^+_\mu \\ A^0_\mu \\ A^-_\mu \end{pmatrix} ,$$

one can check that

$$Y^{\mathrm{T}}_\mu C_q Y_\mu = (\mu/q^2) L_\mu , \tag{27}$$

where

$$C_q = \frac{1}{q^2} \begin{pmatrix} 0 & 0 & 1/q \\ 0 & \mu & 0 \\ q & 0 & 0 \end{pmatrix} .$$

In addition, the action (17) preserves this quadratic form (27), so we can regard C_q as the metric on the q-deformed space of gauge potentials.

Note that L_μ is the Casimir operator of the quadratic algebra (19), i.e.,

$$[L_\mu, A^i_\mu] = 0$$

for any i.

Thus we find the connection:

$$\mathrm{Tr}_q\, \mathbf{A}^2_\mu = Y^{\mathrm{T}}_\mu C_q Y_\mu .$$

4. The exponential map for $SO_{q^2}(3,\mathbb{C})$

Since the exponential maps for $SU_q(2)$ and $SL_q(2)$ are known [6,17], the above mentioned embedding of $SL_q(2)$ in $SO_{q^2}(3,\mathbb{C})$ gives a hint how to find the exponential map for $SO_{q^2}(3,\mathbb{C})$.

Let g be an element of $SL_q(2,\mathbb{C})$ and L be a matrix of the form

$$L = \begin{pmatrix} l_0 & l_+ \\ l_- & kl_0 \end{pmatrix},$$

where k is an arbitrary complex parameter. A pair of g and L is called a (g,L)-pair if the following relations are satisfied:

$$\begin{array}{lll} l_0 a = -(1/k) a l_0, & l_+ a = (1/q) a l_+, & l_- a = (1/q) a l_-, \\ l_0 c = -(1/k) c l_0, & l_+ c = (1/q) c l_+, & l_- c = (1/q) c l_-, \\ l_0 b = -(1/k) b l_0, & l_+ b = q b l_+, & l_- b = q b l_-, \\ l_0 d = -(1/k) d l_0, & l_+ d = q d l_+, & l_- d = q d l_-. \end{array} \tag{28}$$

Define the quantum superplane by the relations:

$$l_0^2 = 0, \quad l_+^2 = 0, \quad l_-^2 = 0, \tag{29}$$

$$l_- l_0 + k l_0 l_- = 0, \quad l_0 l_+ + k l_+ l_0 = 0, \quad l_- l_+ = q^2 l_+ l_-.$$

We know that for a given pair (g,L) and matrix L parametrized by elements of the quantum superplane the element $g\mathrm{e}^{tL}$ for any real t also belongs to $SL_q(2)$ and has the form

$$g\mathrm{e}^{tL} = \tag{30}$$

$$\times \begin{pmatrix} a(t + \frac{1}{6}t^3 l_+ l_-) l_0 & t a l_+ + k b (t + \frac{1}{6}t^3 l_- l_+) l_0 \\ \quad + a(1 + \frac{1}{2}t^2 l_+ l_-) + t b l_- & \quad + b(1 + \frac{1}{2}t^2 l_- l_+) \\ c(t + \frac{1}{6}t^3 l_+ l_-) l_0 & t c l_+ + k d (t + \frac{1}{6}t^3 l_- l_+) l_0 \\ \quad + c(1 + \frac{1}{2}t^2 l_+ l_-) + t d l_- & \quad + d(1 + \frac{1}{2}t^2 l_- l_+) \end{pmatrix}.$$

Proposition 4. *If $P \in SO_{q^2}(3,\mathbb{C})$ is the matrix (25) and the parameter k in the relations (28), (29) equals -1, then the element $P(t) = A(t)P$ with*

$$A(t) = \mathrm{e}^{tS}, \tag{31}$$

$$S = \begin{pmatrix} 2l_0 & -\sqrt{\mu} l_- & 0 \\ -\sqrt{\mu} q l_+ & 0 & (1/q)\sqrt{\mu} l_- \\ 0 & \sqrt{\mu} l_+ & -2l_0 \end{pmatrix}, \tag{32}$$

also belongs to $SO_{q^2}(3,\mathbb{C})$.

Proof. Substituting the entries of $g\mathrm{e}^{tL}$ from (30) in the matrix P and using the commutation relations (28) we get some matrix $A(t)$ which in the case $k = -1$ has the form (30). Let us fix $k = -1$; then the preservation of the additional relations (21) may be shown explicitly by using the formula

$$P(t)^{\mathrm{T}} = (\mathrm{e}^{tS} P)^{\mathrm{T}} = P^{\mathrm{T}} \mathrm{e}^{tS'},$$

where

$$S' = -C^{-1}SC\,, \qquad C = \begin{pmatrix} 0 & 0 & 1/q \\ 0 & 1 & 0 \\ q & 0 & 0 \end{pmatrix}.$$

Then for any real t the additional relations are satisfied:

$$P(t)CP(t)^{\mathrm{T}}C^{-1} = \mathrm{e}^{tS}PCP^{\mathrm{T}}\mathrm{e}^{tS'}C^{-1} = \mathrm{e}^{tS}\mathrm{e}^{tCS'C^{-1}} = 1.$$

Now we see that the exponential map for a generic element of $SO_{q^2}(3,\mathbb{C})$ may be constructed in the following way. Let us identify the entries of the matrix P with those of $g \in SO_{q^2}(3,\mathbb{C})$ (see appendix C) and define the matrix S by (32). Let us say that an (S,g)-pair is given if the commutation relations between the entries of S and g are just the same as between the entries of S and P, namely,

$$[g_{ij}, l_0] = 0\,, \qquad [g_{1i}, l_\pm] = 0\,,$$

$$[g_{0i}, l_\pm]_{q^{\pm 2}} = 0\,, \qquad [g_{2i}, l_\pm]_{(1/q^2)^\pm} = 0\,.$$

Here we have made use of the q^2-commutator $[A,B]_{q^2} = AB - q^2BA$. If the elements of S belong to the quantum superplane, i.e., the relations (29) are satisfied ($k = -1$), then for any real t the element $\mathrm{e}^{tS}g$ also belongs to $SO_{q^2}(3,\mathbb{C})$. Straightforward calculations complete the proof. □

By rescaling the operators l_0, l_+, l_-, we find the connection of the matrix S with the q-deformation of the three-dimensional representation of su(2) ≈ so(3):

$$S = \begin{pmatrix} l_0 & l_- & 0 \\ ql_+ & 0 & (1/q)l_- \\ 0 & -l_+ & -l_0 \end{pmatrix} = l_0T_0 + l_+T_- + l_-T_+\,.$$

The matrices

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1/q \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_- = \begin{pmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

represent the q-deformation of the three-dimensional representation of su(2).

5. Concluding remarks

In this article we have endeavoured to develop the idea that a gauge theory can undergo a q-deformation. One can ask if is it any use trying? By q-deforming a gauge theory we aim to construct physically reasonable theories. In spite of the impressive success of applications of gauge group ideas in the theory of elementary particles there are several open problems in the theories based on the standard Yang–Mills gauge fields. Quark confinement in QCD, the ad hoc introduction of Higgs scalars and the difficulties with grand unification are striking examples of such problems. One can think that these problems are not only technical but may be caused by the rigid framework of gauge theory based on Lie groups. One can hope that some modification of the notion of gauge fields may help solve these problems.

Gauge fields are associated usually with local gauge invariance. The model considered in this note gives an example of a quantum group field theory with global invariance.

As we mentioned before the main obstacle in constructing local quantum group gauge theory is to give a meaning to the local gauge transformations

$$A \to A' = gAg^{-1} + g\partial g^{-1}, \tag{33}$$

since it is necessary to add two elements belonging the Lie algebra analog of a quantum group. In general this Lie algebra analog is not an algebra and does not possess an addition. However, one can ignore this problem and assume that the potentials belong to some algebra B and only the stress tensor $F_{\mu\nu}$ belongs to some minimal set which is invariant under gauge transformations. From this point of view the Lagrangian (6) with A_μ being the subject of the q-deformed Euclidean space (22) is admissible.

The model (6) has the following distinguished features:

(i) it has the right classical limit, $q \to 1$, reproducing the standard Yang–Mills theory;

(ii) since $\mathrm{tr}_q F_{\mu\nu} \neq 0$ the theory has two coupling constants and the second coupling constant drops out when $q = 1$.

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Appendix A. The commutation relations for $SU_q(2)$

$$aa^* + q^2c^*c = 1, \tag{A.1}$$

$$a^*a + cc^* = 1, \tag{A.2}$$

$$ca^* = qa^*c, \tag{A.3}$$

$$ac = qca, \tag{A.4}$$

$$c^*c = cc^*. \tag{A.5}$$

Appendix B. The commutation relations for $GL_q(2)$

A matrix g of the form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an element of $GL_q(2)$ $(q \in \mathbb{C})$ if the following commutation relations are satisfied:

$$ab = qba, \qquad ac = qca, \qquad ad - da = (q - 1/q)bc,$$
$$bc = cb, \qquad bd = qdb, \qquad cd = qdc.$$

Appendix C. List of commutation relations for the quantum group $SO_q(3, \mathbb{C})$

A matrix g of the form

$$g = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix}$$

is an element of $GL_q(2)$ $(q \in \mathbb{C})$ if the following commutation relations are satisfied:

$$g_{00}g_{01} = qg_{01}g_{00}, \qquad g_{01}g_{02} = qg_{02}g_{01},$$
$$g_{00}g_{02} = q^2g_{02}g_{00}, \qquad g_{01}g_{10} = g_{10}g_{01},$$
$$g_{00}g_{10} = qg_{10}g_{00}, \qquad g_{01}g_{20} = qg_{20}g_{01},$$
$$g_{00}g_{20} = q^2g_{20}g_{00}, \qquad g_{10}g_{02} = qg_{02}g_{10},$$
$$g_{02}g_{11} = g_{11}g_{02}, \qquad g_{10}g_{20} = qg_{20}g_{10},$$
$$g_{02}g_{21} = qg_{21}g_{02}, \qquad g_{11}g_{20} = g_{20}g_{11},$$
$$g_{02}g_{20} = g_{20}g_{02}, \qquad g_{20}g_{12} = qg_{12}g_{20},$$
$$g_{02}g_{12} = qg_{12}g_{02}, \qquad g_{12}g_{21} = g_{21}g_{12},$$
$$g_{02}g_{22} = q^2g_{22}g_{02}, \qquad g_{12}g_{22} = qg_{22}g_{12},$$
$$g_{20}g_{21} = qg_{21}g_{20}, \qquad g_{20}g_{22} = q^2g_{22}g_{20},$$
$$g_{21}g_{22} = qg_{22}g_{21},$$

$$(1/q)g_{00}g_{21} - g_{21}g_{00} = (q - 1/q)g_{20}g_{01},$$
$$g_{00}g_{12} - qg_{12}g_{00} = (q - 1/q)g_{10}g_{02},$$
$$g_{01}g_{11} - g_{11}g_{01} = -(1/\sqrt{q})(q - 1/q)g_{10}g_{02},$$
$$g_{00}g_{11} - g_{11}g_{00} = (q - 1/q)g_{01}g_{10},$$
$$g_{01}g_{12} - g_{12}g_{01} = (q - 1/q)g_{11}g_{02},$$
$$g_{21}g_{01} - (1/q)g_{01}g_{21} = (1/\sqrt{q})(q - 1/q)g_{20}g_{02},$$
$$(1/q)g_{01}g_{22} - g_{22}g_{01} = (q - 1/q)g_{21}g_{02},$$
$$g_{11}g_{10} - g_{10}g_{11} = (1/\sqrt{q})(q - 1/q)g_{01}g_{20},$$

$$g_{10}g_{21} - g_{21}g_{10} = (q - 1/q)g_{11}g_{20},$$
$$(1/q)g_{10}g_{22} - g_{22}g_{10} = (q - 1/q)g_{12}g_{20},$$
$$g_{12}g_{11} - g_{11}g_{12} = (1/\sqrt{q})(q - 1/q)g_{02}g_{21},$$
$$g_{21}g_{11} - g_{11}g_{21} = (1/\sqrt{q})(q - 1/q)g_{20}g_{12},$$
$$(1/q)g_{10}g_{22} - g_{22}g_{10} = (q - 1/q)g_{12}g_{20},$$
$$g_{11}g_{22} - g_{22}g_{11} = (q - 1/q)g_{12}g_{21}.$$

The missing commutation relations between the elements of $SO_q(3,\mathbb{C})$ may be found by combining the above ones with the ones following from the additional equations (22).

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